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# Seven-body central configurations

A family of central configurations in the spatial  
seven-body problem

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**Abstract** The main result of this paper is the existence of a new family of central configurations in the Newtonian spatial seven-body problem. This family is unusual in that it is a *simplex stacked central configuration*, i.e the bodies are arranged as concentric three and two dimensional simplexes.

**Keywords** Central configurations · relative equilibria · N-body problem · celestial mechanics.

**Mathematics Subject Classification (2000)** 70F15 · 70F10

## 1 Introduction

The Newtonian  $n$ -body problem concerns the motion of  $n$  particles with masses  $m_i \in \mathbb{R}^+$  and positions  $q_i \in \mathbb{R}^d$ , where  $i = 1, \dots, n$ . The motion is governed by Newton's law of motion

$$\ddot{q}_i + \gamma_i = 0 \quad (1)$$

where

$$\gamma_i = \sum_{k \neq i} m_k R_{ik}(q_i - q_k) \quad (2)$$

and  $R_{ik} = \|q_i - q_k\|^{-3}$ . The *center of mass*  $q_G$  of the system is defined by the relation  $\sum m_i(q_i - q_G) = 0$ .

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The  $n$  particles form a *central configuration* if there exists a  $\lambda \in \mathbb{R}$  such that  $\gamma_i = \lambda(q_i - q_G)$  (Wintner 1941).

Several aspects of the  $n$ -body problem motivate the study of central configurations. First of all, for suitable initial velocities, central configurations may define a *homothetic* motion (i.e. a motion where only the size of the configuration can change) or, in more generality, a *homographic* motion (a motion where the configuration can rotate and change size). In the case of a strictly three-dimensional motion homographic motions that are not homothetic are forbidden. In the planar  $n$ -body problem, instead, there exist homographic motions which are *relative equilibria* (i.e. motions where the configuration rotates but does not change size).

Secondly, for the  $n$ -body problem, every motion starting or ending in total collision is asymptotic to an homothetic motion and has a central configuration as normalized limit (Saari 1980).

Finally, central configurations appear as a key point when studying the topological changes of the integral manifolds. The integral manifold  $I_{hc}$  of the  $n$ -body problem is the set of points having energy  $h$  and angular momentum  $c$ . Changes in the topology of the integral manifold are caused by central configurations. Topological informations on the integral manifold may provide qualitative informations about the motion of the bodies. For further details and references see Albouy (1993).

In this paper we will exhibit a new family of spatial central configurations in the seven body problem. This family is unusual in that it is a *simplex stacked central configuration*, namely a configuration formed by an equilateral triangle contained in a tetrahedron (see figure 1). Examples of trivial *stacked central configurations* (i.e. configurations where a subset of the points is a central configuration) are well known. For instance the configurations where  $N$  particles lie at the vertices of a regular  $N$ -gon are stacked central configurations when  $N$  is not a prime number. Other simple examples are given by the “rosette central configurations”, i.e. planar central configurations where  $N$  particles of mass  $m_1$  lie at the vertices of a  $N$ -gon,  $N$  particles of masses  $m_2$  lie at the vertices of another  $N$ -gon rotated an angle of  $\pi/n$  from the other, and a particle of mass  $m_0$  lies at the center of the two  $N$ -gons (see Sekiguchi (2004) and Lei and Santoprete (2006)).

The first nontrivial example of stacked central configuration, where two bodies on a line are contained in an equilateral triangle, was found only recently (see Hampton (2005)). The new family of central configurations discussed in the present paper seems to be the natural generalization of the five body one found in Hampton (2005).

The results of this paper and those of Hampton (2005) suggest that such families of stacked central configurations might generalize to higher dimensions, with  $n + m + 2$  bodies in  $\mathbb{R}^n$  arranged as concentric  $n$ - and  $m$ -dimensional simplices (for  $m < n$ ). However, we have been unable to find a proof that would easily generalize.

## 2 Central Configurations in $\mathbb{R}^d$

We now want to express the equations for central configurations in a way that is more convenient for our purposes. From equation (2) it is easy to obtain

$$\gamma_i - \gamma_j = (m_i + m_j)R_{ij}(q_i - q_j) + \sum_{k \neq i, j} m_k(R_{ik}(q_i - q_k) - R_{jk}(q_j - q_k)). \quad (3)$$

On the other hand the equation for the central configuration can be expressed as  $\gamma_i - \gamma_j = \lambda(q_i - q_j)$ . Taking the wedge product of equation (3) with the vector  $q_i - q_j$  we get

$$\sum_{k \neq i, j} m_k(R_{ik} - R_{jk})\Lambda_{ijk} = 0 \quad (4)$$

where  $\Lambda_{ijk} = (q_i - q_j) \wedge (q_i - q_k)$ . For a non-collinear configuration, the system of equations (4) says that  $\gamma_i - \gamma_j$  and  $q_i - q_j$  are linearly dependent and therefore is equivalent to the definition of central configuration. The equations (4), in the particular case of a planar central configuration, are known as the Laura-Andoyer equations and the bivector  $\Lambda_{ijk}$  is simply twice the oriented area of the triangle  $(q_i, q_j, q_k)$ . G. Meyer (Meyer 1933) generalized the Laura-Andoyer equations to higher dimensions (see also Hagihara 1970; Albouy 2003).

Equation (4) can be written in a slightly different way if one takes the wedge product of equation (2), (with the indexes renamed:  $i = i_1, j = i_2$ ) with  $(q_{i_2} - q_{i_3}) \wedge (q_{i_3} - q_{i_4}) \wedge \dots \wedge (q_{i_{d-1}} - q_{i_d})$ . Thus the equations for the central configurations in  $\mathbb{R}^d$  can be written as

$$\sum_{k \neq i_1, \dots, i_d} m_k(R_{i_1 k} - R_{i_2 k})\Delta_{i_1, i_2, \dots, i_d, k} = 0 \quad (5)$$

where  $\Delta_{i_1, i_2, \dots, i_d, k} = \det(q_{i_1} - q_{i_2}, \dots, q_{i_{d-1}} - q_{i_d}, q_{i_d} - q_k)$  i.e. it is  $d!$  times the signed volume of the  $d$ -dimensional simplex formed by the masses  $m_{i_1}, m_{i_2}, \dots, m_{i_d}, m_k$ .

In particular, when  $d = 3$ , the equations above can be written as

$$f_{ijh} = \sum_{k \neq i, j, h} m_k(R_{ik} - R_{jk})\Delta_{ijhk} = 0 \quad (6)$$

where  $\Delta_{ijhk}$  is the coefficient of  $(q_i - q_j) \wedge (q_j - q_h) \wedge (q_h - q_k)$ , i.e. it is six times the signed volume of the tetrahedron formed by  $m_i, m_j, m_h, m_k$ . Moreover, since  $f_{ijh} = -f_{jih}$ , the system of equations (6) provides  $n(n-1)(n-2)/2$  equations. Usually the  $\Delta_{ijhk}$  will be replaced by the non-negative volumes  $D_{ijhk}$  in order to make the sign of the terms in our equations more apparent.

*Remark 1* If  $n = d + 1$ , and all the mutual distances  $r_{ij}$  are equal, that is, when all masses form a regular simplex of dimension  $d$ , then condition (4) is verified. Thus the regular simplex of dimension  $d$  is a central configuration in  $\mathbb{R}^d$  for any value of the masses.

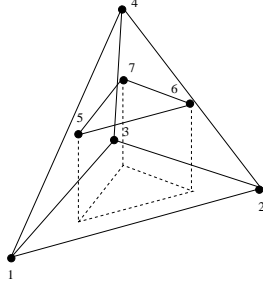


Fig. 1 A typical configuration

### 3 Stacked Central Configurations in $\mathbb{R}^3$

We now investigate configurations with the following symmetries and relations:

$$\begin{aligned}
 r_{12} &= r_{13} = r_{23} = r_{14} = r_{24} = r_{34} = 1 \\
 r_{15} &= r_{26} = r_{37} \\
 r_{45} &= r_{46} = r_{47} \\
 r_{56} &= r_{67} = r_{57} \\
 r_{16} &= r_{17} = r_{25} = r_{27} = r_{35} = r_{36}.
 \end{aligned} \tag{7}$$

Figure 1 depicts a typical configuration. These symmetries result in many of the  $D_{ijkl}$  being equal. These equalities will be exploited to simplify a given  $D_{ijkl}$  to  $D_{mnop}$  where  $\{mnop\} = \{ijkl\}$  and  $m < n < o < p$ ; e.g.  $D_{1475}$  will be replaced by  $D_{1457}$ .

We want to prove the following

**Theorem 1** *For every  $r_{15} \in (0, \sqrt{6}/4)$  there are positive masses  $m_1 = m_2 = m_3, m_4$  and  $m_5 = m_6 = m_7$  and distances which satisfy equation (7) with  $1 > r_{16} > r_{45} > r_{15}$  such that the seven points with these masses form a spatial central configuration.*

In order to prove the theorem we can start observing that, since the configuration is highly symmetric, there are only two degrees of freedom, which can be parametrized with  $r_{15}$  and  $r_{16}$ . Let  $s_{ij} = r_{ij}^2$ , then the other two distances are determined by

$$r_{56} = r_{16}^2 - r_{15}^2 = s_{16} - s_{15}$$

(that indicates that the polygon of vertices  $\{1, 5, 6, 2\}$  is an isosceles trapezoid) and  $g(s_{15}, s_{16}, s_{45}) :=$

$$-3 + 2s_{15} - 3s_{15}^2 + 4s_{16} + 4s_{15}s_{16} - 4s_{16}^2 + 2s_{45} + 2s_{15}s_{45} + 4s_{16}s_{45} - 3s_{45}^2 = 0. \tag{8}$$

The last equation follows from imposing that the 4-volume of the pentachoron (4-simplex) of vertices  $\{1, 2, 3, 4, 5\}$  is zero. This can be done setting the following Cayley-Menger determinant (see Sommerville 1958) to zero:

$$-9216V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & s_{12} & s_{13} & s_{14} & s_{15} \\ 1 & s_{21} & 0 & s_{23} & s_{24} & s_{25} \\ 1 & s_{31} & s_{32} & 0 & s_{34} & s_{35} \\ 1 & s_{41} & s_{42} & s_{43} & 0 & s_{45} \\ 1 & s_{51} & s_{52} & s_{53} & s_{54} & 0 \end{vmatrix} \quad (9)$$

where the  $s_{ij}$  are computed for the above configuration and  $V$  is the 4-volume.

Recall that, where it is convenient, we replace the  $\Delta_{ijhk}$  by the non-negative volumes  $D_{ijhk}$ . However in order to do that we need  $\Delta_{1457} \leq 0$ , since this oriented volume can have different signs for the configuration defined by the equations (7). This corresponds to requiring that the triangle 567 is inside the convex hull of the regular tetrahedron.

In the following we will look for the configurations satisfying  $1 > r_{16} > r_{45} > r_{15}$ . Since the triangle 567 collapses to a point at  $r_{15} = \sqrt{6}/4$ , we restrict our attention to  $r_{15} \in (0, \sqrt{6}/4)$ .

Let us condense the above constraints on the configuration into a region:

**Definition 1**  $\Omega \subset \mathbb{R}^3$  is the set

$$\{(r_{15}, r_{16}, r_{45}) \mid r_{15} \in (0, \sqrt{6}/4), \Delta_{1457} \leq 0, g(r_{15}^2, r_{16}^2, r_{45}^2) = 0, \\ 1 > r_{16} > r_{45} > r_{15}\}$$

Note that  $\Omega$  can be written as a graph over its projection in the  $(r_{15}, r_{16})$  plane, and that is usually how we will think of it. In particular,  $r_{45}$  can be expressed explicitly as a function of  $r_{15}$  and  $r_{16}$ . In a slight abuse of notation, we may refer to seven-body configurations ‘in  $\Omega$ ’, by which we mean a configuration whose  $r_{ij}$  satisfy equation (7), equation (8), and whose  $(r_{15}, r_{16}, r_{45})$  are in  $\Omega$ .

By assumption we will have  $r_{15} \neq r_{16}$ ,  $r_{15} \neq r_{17}$ . From equation (6), with  $i = 1, j = 2, h = 3$ , we obtain

$$f_{123} = -m_5(R_{15} - R_{25})D_{1235} - m_6(R_{16} - R_{26})D_{1236} = 0 \quad (10)$$

and since  $R_{15} = R_{26}$ ,  $R_{16} = R_{25}$  and  $D_{1235} = D_{1236}$  we have

$$(m_5 - m_6)(R_{15} - R_{25})D_{1235} = 0 \quad (11)$$

and we find  $m_5 = m_6$ . Similarly with  $i = 2, j = 3, h = 1$  we find  $m_6 = m_7$ . Thus  $m_5 = m_6 = m_7$ .

Also from equation (6), with  $i = 5, j = 6$  and  $h = 7$ , we obtain

$$f_{567} = -m_1(R_{51} - R_{61})D_{1567} - m_2(R_{52} - R_{62})D_{2567} = 0 \quad (12)$$

and since  $D_{2567} = D_{1567}$  we have

$$(m_1 - m_2)(R_{51} - R_{61})D_{1567} = 0 \quad (13)$$

and thus  $m_1 = m_2$ . Similarly with  $i = 6, j = 7, h = 5$  we find  $m_2 = m_3$ . Thus  $m_1 = m_2 = m_3$ .

Now let  $i = 1$ ,  $j = 4$  and  $h = 7$ . Then from equation (6) we obtain

$$f_{147} = m_5((R_{15} - R_{45})\Delta_{1457} + (R_{16} - R_{45})\Delta_{1467}) = 0 \quad (14)$$

As long as  $m_5 \neq 0$  the previous equation gives a relatively simple constraint on the geometry of our configuration:

$$H := (R_{15} - R_{45})D_{1456} - (R_{45} - R_{16})D_{1467} = 0. \quad (15)$$

Thus we are interested in the subset of  $\Omega$  where  $H = 0$ :

**Definition 2** Let  $\Omega_H \in \Omega$  denote the subset of  $\Omega$  for which  $H = 0$ .

Before analyzing  $H$  in more detail we prove two technical lemmas. Let  $1/a$  be the distance between the plane  $P$  defined by the points 1, 2 and 3 and the plane  $P'$  defined by the points 5, 6 and 7, and let  $b = r_{56}$ . With this notation we can prove the following

**Lemma 1** *Given a configuration in  $\Omega$  we have  $f_{147} = bf_{241}$ ,  $f_{175} = -bf_{251}$ ,  $f_{645} = bf_{461}$  and  $f_{275} = b(f_{251} + f_{271})$ .*

*Proof* To prove that  $f_{147} = -bf_{241}$  note that

$$\begin{aligned} f_{241} &= m_5[(R_{16} - R_{45})(\Delta_{1245} + \Delta_{1247}) + (R_{15} - R_{45})\Delta_{1245}] \\ f_{147} &= -m_5[(R_{16} - R_{45})\Delta_{1467} + (R_{15} - R_{45})\Delta_{1457}]. \end{aligned} \quad (16)$$

A simple computation shows that

$$\Delta_{1457} = \frac{\sqrt{3}b(3 - \sqrt{6}a + \sqrt{6}ab)}{18a} = -b\Delta_{1245} \quad (17)$$

$$\Delta_{1467} = -\frac{\sqrt{3}b(-6 + \sqrt{6}ab + 2\sqrt{6}a)}{18a} = -b(\Delta_{1245} + \Delta_{1247}). \quad (18)$$

This concludes the first part of the proof.

To show that  $f_{175} = -bf_{251}$  consider

$$\begin{aligned} f_{251} &= -m_1(1 - R_{16})\Delta_{1235} - m_4(1 - R_{45})\Delta_{1245} \\ &\quad + m_5(R_{16} - R_{56})\Delta_{1257} \\ f_{175} &= -m_1(1 - R_{16})\Delta_{1257} + m_4(1 - R_{45})\Delta_{1456} \\ &\quad + m_5(R_{16} - R_{56})\Delta_{1567}; \end{aligned} \quad (19)$$

then it is easy to find the following relationships between volumes

$$\begin{aligned} \Delta_{1257} &= \sqrt{3}b/(2a) = -b\Delta_{1235} \\ \Delta_{1456} &= -\frac{\sqrt{3}b(3 - \sqrt{6}a + \sqrt{6}ab)}{18a} = b\Delta_{1245} \\ \Delta_{1567} &= -\sqrt{3}b^2/(2a) = -b\Delta_{1257}. \end{aligned} \quad (20)$$

To show that  $f_{645} = bf_{461}$  consider

$$\begin{aligned} f_{461} &= m_1[(1 - R_{15})\Delta_{1245} - (1 - R_{16})\Delta_{1247}] \\ &\quad + m_5(R_{45} - R_{56})(-\Delta_{1456} + \Delta_{1467}) \\ f_{645} &= -\{-m_1[(1 - R_{15})\Delta_{1456} + (1 - R_{16})(\Delta_{1456} + \Delta_{1467})] \\ &\quad + m_5(R_{45} - R_{56})\Delta_{4567}\} \end{aligned} \quad (21)$$

then we have the following relationships

$$\begin{aligned} \Delta_{1456} &= -\frac{\sqrt{3}b(3 - \sqrt{6}a + \sqrt{6}ab)}{18a} = b\Delta_{1245} \\ \Delta_{1456} + \Delta_{1467} &= -\frac{\sqrt{3}b(-3 + \sqrt{6}a + 2\sqrt{6}ab)}{18a} = -b\Delta_{1247} \\ \Delta_{4567} &= \frac{\sqrt{3}b^2(-3 + \sqrt{6}a)}{6a} = b(\Delta_{1456} - \Delta_{1467}). \end{aligned} \quad (22)$$

Finally to prove that  $f_{275} = b(f_{251} + f_{271})$  we consider

$$\begin{aligned} f_{275} &= m_1[(1 - R_{16})\Delta_{1257} + (1 - R_{15})\Delta_{1257}] + m_4(1 - R_{45})\Delta_{1467} \\ &\quad + m_5(R_{15} - R_{56})\Delta_{1567} \end{aligned}$$

and

$$\begin{aligned} f_{251} &= -m_1(1 - R_{16})\Delta_{1235} - m_4(1 - R_{45})\Delta_{1245} + m_5(R_{16} - R_{56})\Delta_{1257} \\ f_{271} &= -m_1(1 - R_{15})\Delta_{1235} - m_4(1 - R_{45})\Delta_{1247} \\ &\quad - m_5(R_{16} + R_{15} - 2R_{56})\Delta_{1257} \end{aligned} \quad (23)$$

Standard computations give the following relationships

$$\begin{aligned} \Delta_{1257} &= \frac{\sqrt{3}b}{2a} = -b\Delta_{1235} \\ \Delta_{1467} &= \frac{\sqrt{3}b(6 - 2\sqrt{6}a - \sqrt{6}ab)}{18a} = -b(\Delta_{1245} + \Delta_{1247}) \\ \Delta_{1567} &= -\frac{\sqrt{3}b^2}{2a} = -b\Delta_{1257} \end{aligned} \quad (24)$$

This completes the proof.

Using the lemma above we have

**Lemma 2** *Given a configuration in  $\Omega_H$  there exists positive masses for which the configuration is a central configuration.*

*Proof* We have seen that, to have  $f_{123} = f_{231} = 0$  and  $f_{567} = f_{675} = 0$ , we must choose  $m_1 = m_2 = m_3$  and  $m_5 = m_6 = m_7$ . Because of the symmetry of the configuration and of the above choice of the masses we have that

$$\begin{aligned}
f_{123} &= f_{124} = f_{125} = f_{126} = f_{127} = f_{132} = f_{134} = f_{135} = f_{136} = f_{137} \\
&= f_{145} = f_{154} = f_{231} = f_{234} = f_{235} = f_{236} = f_{237} = f_{246} = f_{347} \\
&= f_{264} = f_{374} = f_{451} = f_{462} = f_{473} = f_{561} = f_{562} = f_{563} = f_{564} \\
&= f_{567} = f_{571} = f_{572} = f_{573} = f_{574} = f_{576} = f_{671} = f_{672} = f_{673} \\
&= f_{674} = f_{675} = 0
\end{aligned} \tag{25}$$

A first inspection of the equations shows that the remaining functions that must vanish are  $f_{147}$  and  $f_{241}$ ,  $f_{251}$  and  $f_{175}$ ,  $f_{271}$ ,  $f_{461}$  and  $f_{645}$  and  $f_{275}$ . However the equations above are not independent for the highly symmetric configuration under discussion. Inspecting carefully the remaining equations one finds that the remaining functions that must vanish are actually only  $f_{147}$ ,  $f_{251}$ ,  $f_{271}$ ,  $f_{461}$  (see Lemma 1).

Since  $H = 0$ ,  $f_{147}$  vanishes.  $f_{251}$ ,  $f_{271}$  and  $f_{461}$  form a  $3 \times 3$  homogeneous linear system of the form  $Ax = 0$ , where  $x = (m_1, m_4, m_5)$  and in  $\Omega_H$  the matrix  $A$  takes the form

$$\begin{pmatrix}
(1 - R_{16})D_{1235} & (R_{45} - 1)D_{1245} & (R_{16} - R_{56})D_{1257} \\
(1 - R_{15})D_{1235} & (R_{45} - 1)D_{1247} & (2R_{56} - R_{16} - R_{15})D_{1257} \\
(1 - R_{15})D_{1245} + (R_{16} - 1)D_{1247} & 0 & (R_{56} - R_{45})(D_{1456} + D_{1467})
\end{pmatrix} \tag{26}$$

For a nonzero mass solution  $(m_1, m_4, m_5)$  to exist, the determinant of  $A$  must be zero. The fact that the determinant is zero can be proved showing that the third row is a linear combination of the first two. In order to do that, since  $H = 0$ ,  $R_{45}$  can be written as

$$R_{45} = \frac{R_{15}D_{1456} + R_{16}D_{1467}}{D_{1456} + D_{1467}}$$

so

$$(R_{56} - R_{45})(D_{1456} + D_{1467}) = (R_{56} - R_{15})D_{1456} + (R_{56} - R_{16})D_{1467}.$$

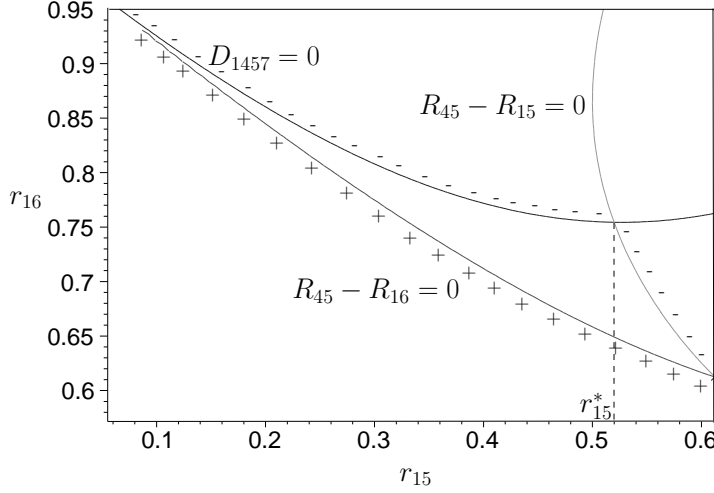
Let  $v_1$ ,  $v_2$  and  $v_3$  be the rows of  $A$ . Then a standard computation shows that  $v_3 = \alpha v_1 + \beta v_2$  where

$$\begin{aligned}
\alpha &= -\frac{-3 + 2\sqrt{6}ab + \sqrt{6}a}{9} = -D_{1247}/D_{1235} \\
\beta &= -\frac{3 + \sqrt{6}ab - \sqrt{6}a}{9} = D_{1245}/D_{1235}.
\end{aligned} \tag{27}$$

This is because

$$\begin{aligned}
&\alpha(1 - R_{45})D_{1245} + \beta(1 - R_{45})D_{1247} \\
&= -(1 - R_{45})D_{1245} \frac{D_{1247}}{D_{1235}} + (1 - R_{45})D_{1247} \frac{D_{1245}}{D_{1235}} = 0.
\end{aligned} \tag{28}$$





**Fig. 2** The curves  $D_{1457} = 0$  and  $R_{45} = R_{16}$  in the plane  $(r_{15}, r_{16})$ . In  $\Omega$   $H < 0$  along  $D_{1456} = 0$  and  $H > 0$  along  $R_{45} = R_{16}$ . The points above the curve  $D_{1457} = 0$  are not in  $\Omega$ .

Moreover

$$\begin{aligned} (R_{56} - R_{15})D_{1257}\beta + (R_{56} - R_{16})(D_{1257}\beta - D_{1257}\alpha) \\ = (R_{56} - R_{15})D_{1456} + (R_{56} - R_{16})D_{1467} \end{aligned} \quad (29)$$

since

$$D_{1257}\beta = -\frac{\sqrt{3}b(3 - \sqrt{6}a + \sqrt{6}ab)}{18a} = D_{1456} \quad (30)$$

$$D_{1257}\beta - D_{1257}\alpha = \frac{\sqrt{3}b(-6 + 2\sqrt{6}a + \sqrt{6}ab)}{18a} = D_{1467} \quad (31)$$

Finally the fact that the masses may be chosen all positive is a consequence of the sign pattern of the coefficient matrix. For the configurations in  $\Omega_H$  the sign pattern is

$$\begin{pmatrix} - & + & - \\ - & + & * \\ - & 0 & + \end{pmatrix} \quad (32)$$

where  $*$  denotes a sign that is unimportant for the discussion. The first and last row imply that all the components of a nonzero nullvector have the same sign.

To complete the proof of the theorem we only have to show that for every  $r_{15} \in (0, \sqrt{6}/4)$  we can choose  $r_{16}$  and  $r_{45}$  so that  $H = 0$  and  $(r_{15}, r_{16}, r_{45}) \in \Omega$ . Note that in  $\Omega$ ,  $H$  is negative when  $D_{1456} = 0$  (i.e.  $D_{1457} = 0$ , since  $D_{1457} = D_{1456}$ ) and when  $R_{45} = R_{15}$ .  $H$  is positive when  $R_{45} = R_{16}$ . For a given value of  $0 < r_{15} < r_{15}^*$  one can vary  $r_{16}$  from the curve  $R_{45} = R_{16}$  to the curve  $D_{1457} = 0$  (Figure 2). Thus  $H = 0$  is satisfied for some value of

$r_{16}$ . For  $r_{15}^* \leq r_{15} < \sqrt{6}/4$  (i.e. when the corresponding point on the curve  $R_{45} - R_{15}$  is in  $\Omega$ ) one can vary  $r_{16}$  from the curve  $R_{45} = R_{16}$  to the curve  $R_{45} - R_{15} = 0$ . Hence  $H = 0$  for some value of  $r_{16}$ .

Therefore there is at least one central configuration for every  $r_{15} \in (0, \sqrt{6}/4)$ .

We can show a little more:

**Theorem 2** *There is only one such configuration for each  $r_{15} \in (0, \sqrt{6}/4)$ .*

*Proof* To prove that there is only one such configuration for each  $r_{15}$  it is enough to show that  $H' < 0$ , where the prime denotes differentiation with respect to  $r_{16}$  and with  $r_{15}$  held fixed. From equation (8), using implicit differentiation, we have

$$r'_{45} = -\frac{2r_{16}(1 + r_{15}^2 - 2r_{16}^2 + r_{45}^2)}{r_{45}(1 + r_{15}^2 + 2r_{16}^2 - 3r_{45}^2)} \quad (33)$$

which is negative for the configurations in  $\Omega$ .

Recall that  $H = (R_{15} - R_{45})D_{1456} - (R_{45} - R_{16})D_{1467}$ . Now consider the derivative

$$\begin{aligned} H' &= 3r_{45}^{-4}r'_{45}D_{1456} + (R_{15} - R_{45})D'_{1456} + (3r_{45}^{-4}r'_{45} - 3r_{16}^{-4})D_{1467} \\ &\quad - (R_{45} - R_{16})D'_{1467} \\ &= \frac{1}{2D_{1456}} (6r_{45}^{-4}r'_{45}D_{1456}^2 + (R_{15} - R_{45})(D_{1456}^2)') \\ &\quad + \frac{1}{2D_{1467}} (6(r_{45}^{-4}r'_{45} - r_{16}^{-4})D_{1467}^2 - (R_{45} - R_{16})(D_{1467}^2)') \end{aligned} \quad (34)$$

The point of the above rearrangements is to get all of the volume expressions into polynomial form. The only parts of this expression whose signs are not clear on the region of interest are the volume derivatives  $(D_{1467}^2)'$  and  $(D_{1456}^2)'$ . Unfortunately, the latter does indeed become negative within  $\Omega$ , and even on the set  $\Omega_H = 0$ .

We will first show that  $(D_{1467}^2)'$  is positive on the set  $\Omega_H$ . Let us write

$$(D_{1467}^2)' = \frac{4r_{16}(r_{16}^2 - r_{15}^2)t_1}{1 + r_{15}^2 + 2r_{16}^2 - 3r_{45}^2},$$

where  $t_1$  is a polynomial in  $r_{15}$ ,  $r_{16}$ , and  $r_{45}$ . The sign is clearly determined by the sign of  $t_1$ . If we compute the remainder  $t_2$  of  $t_1$  divided by the polynomial in equation (8), with respect to the lexicographic monomial ordering  $r_{16} \succ r_{15} \succ r_{45}$ , we find that

$$t_2 = (1 - r_{15}^2 + r_{45}^2) [2(r_{16}^2 - r_{45}^2)(1 - r_{15}^2 + r_{45}^2) + 16D_{145}^2],$$

where  $D_{145}$  is the area of the triangle formed by the points  $q_1$ ,  $q_4$ , and  $q_5$ . This form of the expression is clearly positive.

To rule out a zero of  $H'$  on the set  $H = 0$  we began by computing two Gröbner bases (using Magma) of the following systems. Both were constrained to include the set  $H = 0$ , by using the following two polynomials:

$H$  multiplied by  $(R_{15} - R_{45})D_{1456} + (R_{45} - R_{16})D_{1467}$ , with denominators cleared (denoted below as  $b_1$ ), and the condition (8) converted to the  $r_{ij}$  variables (denoted as  $b_2$ ). Both of these polynomials also possess a number of components irrelevant to our analysis, but this is unavoidable. To exclude components of the solutions of these systems with any of the  $r_{ij} = 0$ , the condition  $r_{15}r_{16}r_{45}w - 1 = 0$  was also added (denoted by  $b_3$ ). These three polynomials will be referred to as the base system:

$$\begin{aligned}
b_1 = & -r_{45}^6 r_{15}^{10} + 2r_{16}^3 r_{45}^3 r_{15}^{10} + r_{16}^8 r_{15}^8 - r_{16}^6 r_{15}^8 + 2r_{16}^2 r_{45}^6 r_{15}^8 - 4r_{16}^5 r_{45}^3 r_{15}^8 - r_{16}^6 r_{45}^2 r_{15}^8 \\
& - 2r_{16}^6 r_{45}^3 r_{15}^7 - r_{16}^{10} r_{15}^6 - r_{45}^{10} r_{15}^6 + r_{16}^8 r_{15}^6 + 2r_{16}^2 r_{45}^8 r_{15}^6 + 2r_{45}^8 r_{15}^6 + 2r_{16}^3 r_{45}^7 r_{15}^6 \\
& - 2r_{16}^4 r_{45}^6 r_{15}^6 + 2r_{16}^2 r_{45}^6 r_{15}^6 - r_{45}^6 r_{15}^6 - 4r_{16}^5 r_{45}^5 r_{15}^6 - 4r_{16}^3 r_{45}^5 r_{15}^6 + 4r_{16}^7 r_{45}^3 r_{15}^6 \\
& - 4r_{16}^5 r_{45}^3 r_{15}^6 + 2r_{16}^3 r_{45}^3 r_{15}^6 + r_{16}^8 r_{45}^2 r_{15}^6 + r_{16}^6 r_{45}^2 r_{15}^6 + 2r_{16}^6 r_{45}^5 r_{15}^5 + 2r_{16}^8 r_{45}^3 r_{15}^5 \\
& + 2r_{16}^6 r_{45}^3 r_{15}^5 + r_{16}^6 r_{45}^6 r_{15}^4 - 2r_{16}^6 r_{45}^7 r_{15}^3 + 2r_{16}^8 r_{45}^5 r_{15}^3 + 2r_{16}^6 r_{45}^5 r_{15}^3 - 2r_{16}^{10} r_{45}^3 r_{15}^3 \\
& + 2r_{16}^8 r_{45}^3 r_{15}^3 - 2r_{16}^6 r_{45}^3 r_{15}^3 - r_{16}^6 r_{45}^8 r_{15}^2 - r_{16}^8 r_{45}^6 r_{15}^2 - r_{16}^6 r_{45}^6 r_{15}^2 + r_{16}^6 r_{45}^{10} \\
& - r_{16}^8 r_{45}^8 - r_{16}^6 r_{45}^8 + r_{16}^{10} r_{45}^6 - r_{16}^8 r_{45}^6 + r_{16}^6 r_{45}^6, \\
b_2 = & 3r_{15}^4 - 4r_{16}^2 r_{15}^2 - 2r_{45}^2 r_{15}^2 - 2r_{15}^2 + 4r_{16}^4 + 3r_{45}^4 - 4r_{16}^2 - 4r_{16}^2 r_{45}^2 - 2r_{45}^2 + 3, \\
b_3 = & r_{15}r_{16}r_{45}w - 1
\end{aligned} \tag{35}$$

The first system also included the sign-determining factor of  $6r_{45}^{-4}r_{45}'D_{1456}^2 + (R_{15} - R_{45})(D_{1456}^2)'$ , a polynomial  $p_1$  of total degree 11 in the variables  $r_{15}$ ,  $r_{16}$ , and  $r_{45}$ <sup>1</sup>:

$$\begin{aligned}
p_1 = & 6r_{15}^{11} - 24r_{16}^2 r_{15}^9 - r_{45}^2 r_{15}^9 + 36r_{16}^4 r_{15}^7 + 8r_{45}^4 r_{15}^7 - 5r_{16}^2 r_{45}^2 r_{15}^7 - 16r_{45}^2 r_{15}^7 \\
& + r_{45}^5 r_{15}^6 - 30r_{16}^6 r_{15}^5 - 3r_{45}^6 r_{15}^5 + 18r_{16}^4 r_{15}^5 - 16r_{16}^2 r_{45}^4 r_{15}^5 - 2r_{45}^4 r_{15}^5 \\
& - 18r_{16}^2 r_{15}^5 + 22r_{16}^4 r_{45}^2 r_{15}^5 + 22r_{16}^2 r_{45}^2 r_{15}^5 + r_{45}^2 r_{15}^5 + 6r_{15}^5 - 8r_{45}^7 r_{15}^4 \\
& + 5r_{16}^2 r_{45}^5 r_{15}^4 - 2r_{45}^5 r_{15}^4 + 12r_{16}^8 r_{15}^3 + 6r_{45}^8 r_{15}^3 - 18r_{16}^6 r_{15}^3 - 15r_{16}^2 r_{45}^6 r_{15}^3 \\
& - 8r_{45}^6 r_{15}^3 + 18r_{16}^4 r_{15}^3 + 26r_{16}^4 r_{45}^4 r_{15}^3 + 8r_{45}^4 r_{15}^3 - 6r_{16}^2 r_{15}^3 - 22r_{16}^6 r_{45}^2 r_{15}^3 \\
& + 4r_{16}^4 r_{45}^3 r_{15}^3 - 3r_{16}^2 r_{45}^2 r_{15}^3 - 2r_{45}^2 r_{15}^3 + 9r_{45}^9 r_{15}^2 - 2r_{16}^2 r_{45}^7 r_{15}^2 + 2r_{45}^7 r_{15}^2 \\
& - 4r_{16}^4 r_{45}^5 r_{15}^2 - 4r_{16}^2 r_{45}^5 r_{15}^2 - r_{45}^5 r_{15}^2 - 6r_{15}^{11} + 9r_{16}^2 r_{45}^9 + 8r_{45}^9 - 8r_{16}^4 r_{45}^7 \\
& - 8r_{45}^7 + 4r_{16}^6 r_{45}^5 - 4r_{16}^4 r_{45}^5 + 3r_{16}^2 r_{45}^5 + 2r_{45}^5
\end{aligned} \tag{36}$$

Since  $H'$  has the form  $-(a + bp_1)$  with  $a, b$  positive functions, if  $p_1$  is positive on  $\Omega_H$  then  $H' < 0$  on  $\Omega_H$ . After eliminating  $w$ , a lexicographic term order with  $r_{45} \succ r_{16} \succ r_{15}$  was used to find a polynomial  $q_1(r_{15})$  of degree 404.

<sup>1</sup> To speed the computation, the actual initial system used in the Groebner basis computation included another polynomial in the ideal generated by  $p_1$  and  $b_2$ , but this is theoretically redundant.

The first and last few terms of  $q_1$  are shown below:

$$\begin{aligned}
q_1 = & r_{15}^{404} - (804495232776247942366788446548795369/ \\
& 6172585442506994000774739535727460)r_{15}^{402} \\
& + (19481733600623784042235202979633272487804463/ \\
& 2759886403053727157626401541214461915200)r_{15}^{400} + \dots \\
& - (139/99902129761265947794751686948283396618716394291200)r_{15}^2 + \\
& (1/66601419840843965196501124632188931079144262860800)
\end{aligned} \tag{37}$$

There are four zeros of  $q_1$  in the interval  $(0, \frac{\sqrt{6}}{4})$ ; a rigorous demonstration of this is possible using interval arithmetic. These zeros are approximately at  $r_1 = .5104\dots$ ,  $r_2 = .5384\dots$ ,  $r_3 = .5774\dots$ , and  $r_4 = .5856\dots$

For the second system, we first eliminated  $D_{1467}$  from the expression of  $H'$  by using the condition that  $H = 0$  rearranged as  $D_{1467} = D_{1456} \frac{R_{15} - R_{45}}{R_{45} - R_{16}}$ . After factoring out the positive quantity  $D_{1456}$  we can write

$$\begin{aligned}
H' = & \frac{1}{2D_{1456}} [6r_{45}^{-4}r'_{45}D_{1456}^2 + (R_{15} - R_{45})(D_{1456}^2)' + \\
& \frac{R_{45} - R_{16}}{R_{15} - R_{45}} (6(r_{45}^{-4}r'_{45} - r_{16}^{-4})D_{1467}^2 - (R_{45} - R_{16})(D_{1467}^2)') ]
\end{aligned}$$

Now we clear denominators from the following partial sum:

$$(R_{15} - R_{45})(D_{1456}^2)' + 6 \frac{R_{45} - R_{16}}{R_{15} - R_{45}} (r_{45}^{-4}r'_{45} - r_{16}^{-4})D_{1467}^2$$

and add the numerator  $p_2$  to our base system. If this polynomial does not vanish on  $\Omega_H$  then neither does  $H'$ . The full polynomial  $p_2$  is:

$$\begin{aligned}
p_2 = & 24r_{15}^6r_{16}^{16} - 60r_{15}^8r_{16}^{14} + 4r_{45}^8r_{16}^{14} - 36r_{15}^6r_{16}^{14} - 8r_{15}^3r_{45}^5r_{16}^{14} - 32r_{15}^6r_{45}^2r_{16}^{14} \\
& - 24r_{15}^6r_{45}^3r_{16}^{13} + 60r_{15}^{10}r_{16}^{12} - 8r_{45}^{10}r_{16}^{12} + 60r_{15}^8r_{16}^{12} - 4r_{15}^2r_{45}^8r_{16}^{12} - 4r_{45}^8r_{16}^{12} \\
& + 16r_{15}^3r_{45}^7r_{16}^{12} + 24r_{15}^6r_{16}^{12} + 8r_{15}^5r_{45}^5r_{16}^{12} + 8r_{15}^3r_{45}^5r_{16}^{12} + 16r_{15}^6r_{45}^4r_{16}^{12} \\
& + 56r_{15}^8r_{45}^2r_{16}^{12} - 4r_{15}^6r_{45}^2r_{16}^{12} + 24r_{15}^6r_{45}^5r_{16}^{11} + 60r_{15}^8r_{45}^3r_{16}^{11} + 36r_{15}^6r_{45}^3r_{16}^{11} \\
& - 30r_{15}^{12}r_{16}^{10} + 9r_{45}^{12}r_{16}^{10} - 30r_{15}^{10}r_{16}^{10} - 2r_{15}^2r_{45}^{10}r_{16}^{10} - 18r_{15}^3r_{45}^9r_{16}^{10} - 30r_{15}^8r_{16}^{10} \\
& + 5r_{15}^4r_{45}^8r_{16}^{10} - 4r_{15}^2r_{45}^8r_{16}^{10} + 3r_{45}^8r_{16}^{10} + 4r_{15}^5r_{45}^7r_{16}^{10} - 6r_{15}^6r_{16}^{10} + 3r_{15}^6r_{45}^6r_{16}^{10} \\
& - 10r_{15}^7r_{45}^5r_{16}^{10} + 8r_{15}^5r_{45}^5r_{16}^{10} - 6r_{15}^3r_{45}^5r_{16}^{10} - 32r_{15}^8r_{45}^4r_{16}^{10} + 6r_{15}^6r_{45}^4r_{16}^{10} \\
& - 25r_{15}^{10}r_{45}^2r_{16}^{10} + 8r_{15}^8r_{45}^2r_{16}^{10} + 9r_{15}^6r_{45}^2r_{16}^{10} + 6r_{15}^6r_{45}^7r_{16}^9 - 42r_{15}^8r_{45}^5r_{16}^9 \\
& + 6r_{15}^6r_{45}^5r_{16}^9 - 60r_{15}^{10}r_{45}^3r_{16}^9 - 60r_{15}^8r_{45}^3r_{16}^9 - 24r_{15}^6r_{45}^3r_{16}^9 + 6r_{15}^{14}r_{16}^8 - 6r_{45}^{14}r_{16}^8 \\
& + 6r_{15}^{12}r_{16}^8 + 9r_{15}^2r_{45}^{12}r_{16}^8 + 8r_{45}^{12}r_{16}^8 + 12r_{15}^3r_{45}^{11}r_{16}^8 + 6r_{15}^{10}r_{16}^8 - 8r_{15}^4r_{45}^{10}r_{16}^8 \\
& + 2r_{15}^2r_{45}^{10}r_{16}^8 - 8r_{45}^{10}r_{16}^8 - 18r_{15}^5r_{45}^9r_{16}^8 - 16r_{15}^3r_{45}^9r_{16}^8 + 6r_{15}^8r_{16}^8 + 7r_{15}^6r_{45}^8r_{16}^8 \\
& - 2r_{15}^4r_{45}^8r_{16}^8 - r_{15}^2r_{45}^8r_{16}^8 + 2r_{45}^8r_{16}^8 + 16r_{15}^7r_{45}^7r_{16}^8 - 4r_{15}^5r_{45}^7r_{16}^8 + 16r_{15}^3r_{45}^7r_{16}^8
\end{aligned}$$

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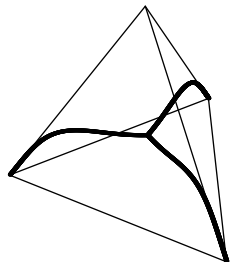

$$\begin{aligned}
& + 15r_{15}^8 r_{45}^6 r_{16}^8 + 8r_{15}^6 r_{45}^6 r_{16}^8 - 2r_{15}^9 r_{45}^5 r_{16}^8 + 4r_{15}^7 r_{45}^5 r_{16}^8 + 2r_{15}^5 r_{45}^5 r_{16}^8 \\
& - 4r_{15}^3 r_{45}^5 r_{16}^8 - 2r_{15}^{10} r_{45}^4 r_{16}^8 - 4r_{15}^8 r_{45}^4 r_{16}^8 - 8r_{15}^6 r_{45}^4 r_{16}^8 + 7r_{15}^{12} r_{45}^2 r_{16}^8 \\
& - 14r_{15}^{10} r_{45}^2 r_{16}^8 - 7r_{15}^8 r_{45}^2 r_{16}^8 + 2r_{15}^6 r_{45}^2 r_{16}^8 - 18r_{15}^6 r_{45}^9 r_{16}^7 - 12r_{15}^8 r_{45}^7 r_{16}^7 \\
& - 6r_{15}^6 r_{45}^7 r_{16}^7 + 24r_{15}^{10} r_{45}^5 r_{16}^7 - 6r_{15}^8 r_{45}^5 r_{16}^7 - 6r_{15}^6 r_{45}^5 r_{16}^7 + 30r_{15}^{12} r_{45}^3 r_{16}^7 \\
& + 30r_{15}^{10} r_{45}^3 r_{16}^7 + 30r_{15}^8 r_{45}^3 r_{16}^7 + 6r_{15}^6 r_{45}^3 r_{16}^7 - 30r_{15}^6 r_{45}^{10} r_{16}^6 - 18r_{15}^8 r_{45}^8 r_{16}^6 \\
& - 6r_{15}^6 r_{45}^8 r_{16}^6 + 9r_{15}^6 r_{45}^{11} r_{16}^5 + 15r_{15}^8 r_{45}^9 r_{16}^5 - 21r_{15}^6 r_{45}^9 r_{16}^5 + 15r_{15}^{10} r_{45}^7 r_{16}^5 \\
& + 12r_{15}^8 r_{45}^7 r_{16}^5 + 15r_{15}^6 r_{45}^7 r_{16}^5 - 9r_{15}^{12} r_{45}^5 r_{16}^5 - 3r_{15}^{10} r_{45}^5 r_{16}^5 + 3r_{15}^8 r_{45}^5 r_{16}^5 \\
& - 3r_{15}^6 r_{45}^5 r_{16}^5 - 6r_{15}^{14} r_{45}^3 r_{16}^5 - 6r_{15}^{12} r_{45}^3 r_{16}^5 - 6r_{15}^{10} r_{45}^3 r_{16}^5 - 6r_{15}^8 r_{45}^3 r_{16}^5 \quad (38) \\
& + 24r_{15}^6 r_{45}^{12} r_{16}^4 + 42r_{15}^8 r_{45}^{10} r_{16}^4 + 6r_{15}^{10} r_{45}^8 r_{16}^4 - 6r_{15}^8 r_{45}^8 r_{16}^4 - 9r_{15}^6 r_{45}^{11} r_{16}^3 \\
& + 3r_{15}^{10} r_{45}^9 r_{16}^3 + 21r_{15}^8 r_{45}^9 r_{16}^3 - 9r_{15}^{12} r_{45}^7 r_{16}^3 - 6r_{15}^{10} r_{45}^7 r_{16}^3 - 15r_{15}^8 r_{45}^7 r_{16}^3 \\
& + 3r_{15}^{14} r_{45}^5 r_{16}^3 + 3r_{15}^{12} r_{45}^5 r_{16}^3 + 3r_{15}^{10} r_{45}^5 r_{16}^3 + 3r_{15}^8 r_{45}^5 r_{16}^3 - 9r_{15}^6 r_{45}^{14} r_{16}^2 \\
& - 21r_{15}^8 r_{45}^{12} r_{16}^2 + 21r_{15}^6 r_{45}^{12} r_{16}^2 - 21r_{15}^{10} r_{45}^{10} r_{16}^2 - 6r_{15}^8 r_{45}^{10} r_{16}^2 - 15r_{15}^6 r_{45}^{10} r_{16}^2 \\
& + 3r_{15}^{12} r_{45}^8 r_{16}^2 + 15r_{15}^{10} r_{45}^8 r_{16}^2 + 3r_{15}^8 r_{45}^8 r_{16}^2 + 3r_{15}^6 r_{45}^8 r_{16}^2 + 9r_{15}^8 r_{45}^{14} \\
& - 3r_{15}^{10} r_{45}^{12} - 21r_{15}^8 r_{45}^{12} + 9r_{15}^{12} r_{45}^{10} + 6r_{15}^{10} r_{45}^{10} + 15r_{15}^8 r_{45}^{10} - 3r_{15}^{14} r_{45}^8 \\
& - 3r_{15}^{12} r_{45}^8 - 3r_{15}^{10} r_{45}^8 - 3r_{15}^8 r_{45}^8
\end{aligned}$$

Again we computed a Gröbner basis, in the same manner as for the first system, and obtained a polynomial  $q_2(r_{15})$  of degree 466 which is in the ideal generated by the polynomials of the second system. The first and last few terms of  $q_2$  are:

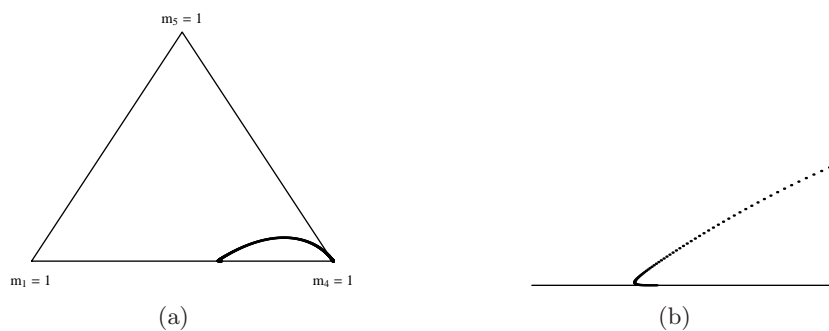
$$\begin{aligned}
q_2 = & r_{15}^{466} - (397970325260732999941673276305737323331716006483/ \\
& 4063279255141998390265984630470563259415746780)r_{15}^{464} \\
& + (717654067077578250690275945137718952732059392991627077/ \\
& 585112212740447768198301786787761109355867536320000)r_{15}^{462} + \dots \\
& + (769475777092319973480299/2698353810706234317867355469145285098 \\
& 555299416887370190209448673280000)r_{15}^{12} \\
& - (2505469532410439724481/8994512702354114392891184897150950328517 \\
& 66472295790063403149557760000)
\end{aligned} \quad (39)$$

Again it is possible to rigorously compute the existence of exactly three roots in the interval  $(0, \frac{\sqrt{6}}{4})$ . These are located at approximately  $r_5 = .5004\dots$ ,  $r_6 = .5027\dots$ , and  $r_7 = .5252\dots$ .

If we consider the arrangement of the roots of the polynomials  $q_1$  and  $q_2$ , we see that we need only verify that on the set  $\Omega_H$ ,  $p_1$  is positive for  $r_{15} \in (0, r_1)$ ,  $p_2$  is positive for  $r_{15} \in (r_7, \frac{\sqrt{6}}{4})$ , and that either  $p_1$  or  $p_2$  is positive for  $r_{15} \in (r_1, r_7)$ . To do this, we can pick three rational values for  $r_{15}$  each of the three intervals just described and compute the Gröbner basis of the base system specialized to those values. We chose to use the values  $\frac{1}{2}$ ,



**Fig. 3** Stacked central configuration family



**Fig. 4** (a) Masses of the configurations. (b) Close up of masses near  $m_5 = 0$ .

$\frac{11}{21}$ , and  $\frac{3}{5}$ . Using interval arithmetic, it is possible to verify the appropriate signs of  $p_1$  and  $p_2$ .

#### 4 Numerical visualizations

Since it is often helpful to have an accurate sketch of phenomena, we reproduce here an image of the family of central configurations described above (Figure 3), and the associated masses (Figure 4). The masses were normalized so that  $m_1 + m_4 + m_5 = 1$  and then projected into that plane.

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## 5 Acknowledgements

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## References

1. Albouy, A.: Integral manifolds of the N-body problem. *Invent. math.* **114**, 463-488 (1993)
2. Albouy, A.: On a Paper of Moeckel On Central Configurations. *Reg. Chaot. Dyn.* **8**, 133-142 (2003)
3. Hagihara, Y.: *Celestial Mechanics*. V 1 chap. 3. The MIT Press, Cambridge, MA (1970)
4. Hampton, M.: Stacked central configurations: new examples in the planar five-body problem. *Nonlinearity*. **18**, 2299-2304 (2005)
5. Lei, J. and Santoprete M.: Rosette central configurations, degenerate central configurations and bifurcations. *Cel. Mech Dyn. Astr.* **94**, 271-287 (2006)
6. Meyer, G: Solutions voisines des solutions de Lagrange dans le probleme des n corps. *Ann. Obs. Bordeaux*. **17**, 77-252 (1933)
7. Saari, D.: On the Role and Properties of n-body Central Configurations. *Celestial Mech.* **21**, 9-20 (1980)
8. Sekiguchi, M: Bifurcations of central configurations in the  $2N+1$  body problem, *Cel. Mech. Dyn. Astr.* **90**, 355-360 (2004)
9. Sommerville, D. *An Introduction to the Geometry of N Dimensions*. Dover, New York (1958)
10. Wintner, A. *The Analytical Foundations of Celestial Mechanics*. Princeton University Press, Princeton NJ (1941)